

Cavity-Optomechanics with Spin-Orbit Coupled Spinor Bose-Einstein Condensate

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Cavity-optomechanics, an exploitation of mechanical-effects of light to couple optical-field with mechanical-objects, has made remarkable progress. Besides, spin-orbit (SO)-coupling, interaction between spin of a quantum-particle and its momentum, has provided foundation to analyze various phenomena like spin-Hall effect and topological-insulators. However, SO-coupling and corresponding topological-features have not been examined in optical-cavity with one vibrational-mirror. Here we report cavity-optomechanics with SO-coupled Bose-Einstein condensate, inducing non-Abelian gauge-field in cavity. We ascertain the influences of SO-coupling and long-range atomic-interactions on low-temperature dynamics which can be experimentally measured by maneuvering area underneath density-noise spectrum. It is detected that not only optomechanical-coupling is modifying topological properties of atomic dressed-states but SO-coupling induced topological-effects are also enabling us to control effective-temperature of mechanical-mirror and dynamic structure factor, which is measurable by detecting neutron-scattering. Our findings are testable in a realistic setup and provide

foundations to manipulate SO-coupling in the field of quantum optics and quantum computation.

Cavity-optomechanics provides splendid foundations in utilizing mechanical effects of light to couple optical degree of freedom with mechanical degree of freedom ¹⁻⁴. A pivotal paradigm was to cool vibrational modes of mechanical degree of freedom to its quantum mechanical ground state ⁵⁻⁸. Interaction of optical field with mechanical-objects in the form of radiation pressure, causes the development of numerous sensors and devices in quantum metrology. The exploitation of mechanical effects of light leads researchers to develop and utilize micro and nano scale mirrors⁹⁻¹³, gravitational wave detectors ¹⁴ and optomechanical crystals ¹⁵. The demonstration of cavity-optomechanics with other physical objects, like ultra-cold gases ¹⁶ and Bose-Einstein condensate (BEC) ¹⁷, opens up various new aspects of research in the study of matter-wave interactions, or more precisely saying, it creates more reliable interface between solid-state physics and optical science. These notable achievements motivate researchers to study bistable-dynamics of cavity-optomechanics with BEC ^{18,19}, high fidelity state transfer ^{20,21}, entanglement among multiple degrees of freedom ²²⁻²⁴, macroscopic tunneling effects in optomechanics ²⁵ and role reversal dynamics between matter-wave and quantized light field. These impressive investigations have assist us in developing better understanding of quantum-classical interface ²⁶. However, spin-orbit interaction and corresponding topological-properties have not been considered and investigated in cavity-optomechanics so far.

Spin-orbit (SO)-coupling, a stunning phenomenon describing interaction between spin of

quantum particle and its momentum, has made remarkable progress in last few years ²⁷. In spite of the fact that atomic intrinsic SO-coupling only affects electronic structure and its strength depends on the intrinsic characteristics of material, but atomic intrinsic SO-coupling cannot couple atomic spin with atomic momentum. However, SO-coupling can be engineered in neutral cold atoms by dressing atomic spin-states with the help of optical lasers ²⁸ producing abelian and non-abelian synthetic gauge field ²⁹. Such artificial coupling is the result of Zeeman interaction between magnetic-moment of particle and applied magnetic field in the frame of particle. When electro-static field $E_s = E_0 \hat{z}$ generates magnetic field $B = (E_0 \hbar / mc^2)(k_x, k_y, 0)$ to the particle moving with momentum $\hbar(k_x, k_y, k_z)$, where c is the speed of light and m is the mass of particle, it will produce Zeeman shift $-\mu \cdot B(k) \propto \sigma_x k_y - \sigma_y k_x$, known as Rashba ³⁰ type SO-coupling. The combination of Rashba and Dresselhaus ³¹ type $(-\sigma_x k_y - \sigma_y k_x)$ will generate two-dimensional SO-coupling. Recent experimental demonstration of SO-coupling in ultra-cold Bose-gases and Fermi-gases ^{32–34} provides a fascinating approach in precisely exploring many-body physics of strongly correlated systems. SO-coupling possesses a significant value in condensed matter physics and it is essential to analyze spin-Hall effect ^{35,36}, topological insulators ^{37–40} and spintronic devices ⁴¹. The SO-coupling in one-dimensional optical lattices ⁴² and optical cavities ^{43–45} demonstrate the significance of this phenomenon and a lot of investigations have been performed in this direction ^{46–54,57}. Although the SO-coupling has been thoroughly investigated theoretically and experimentally even in the cavity environment but SO-coupling and corresponding topological-properties have not been explored in optomechanical cavity, with one vibrational-mirror, in the presence of quantum-noises.

Here we investigate an optomechanical system with one vibrational-mirror (mechanical-

mode) containing SO-coupled BEC with the help of bias magnetic field and two counter-propagating Raman lasers. We explore the dynamics of atomic states as well as mechanical-mode in presence of quantum-noises^{55,58} and topological-features induced by SO-coupling. The two intersecting Raman beams will couple two laser-dressed hyperfine states with synthetic $1/2$ -spin to their motional degree of freedom generating SO-coupling and cause topological variations. These SO-coupled or topological hyperfine states will produce atomic back-action inside the cavity after interacting with intra-cavity field. This atomic back-action will not only modify intra-cavity photon number and mechanical-mode, attached to the system via intra-cavity field, but will also affect atomic dynamics itself, such as reduction of heating-effects caused by Raman lasers^{43,45}. Similarly, cavity induced long-range atom-atom interactions, inside the cavity, will also modify atomic back-action of light. Therefore, we emphasize on the effects of SO-coupling and atom-atom interactions on mechanical-mode as well as atomic-modes, especially on their low-temperature profile by measuring the area underneath density-noise spectrum (DNS)^{58,59}. In a similar way, mechanical oscillator will also show its influence on the atomic dressed-states and its topological-features. Further, we compute dynamic structure factor (DSF) which is another fascinating phenomenon of quantum many-body system related to the emergence of density fluctuations of particles⁶⁰. It has provided a strong foundation to analyze the emergence of quasi-particles as well as measure their excitation energy, lifetime and mean occupation number⁶¹ which is connected to the temperature of the correlated particles. In solid state system, the DSF can be observed by shining neutron beam on the atomic assemble and measuring inelastically scattering of particles with respect to their energy $\hbar\omega$ and momentum $\hbar k$ change, known as neutron scattering. Further, in the case of quantum par-

ticles, DSF can be obtained by determining scattering of interacting photons from atomic-mode having low-temperature profile, known as Bragg spectroscopy^{62–66}. All these particular methods are connected to the linear response of perturbed particles and quasi-particles excitations. However, a different attempt was made to calculate static structure in two-dimensional quantum gas⁶⁷ which is similar to the demonstration of noise correlation functions in ultracold quantum gas^{58,68}. Here we choose a similar method to compute DSF as used for experimental observations in ref.⁶⁹, where they computed DSF by manipulating photons leaking-out from cavity. Moreover, we have provided a particular set of parameters and procedures very close to the present available experimental quests which makes our study of cavity-optomechanics with SO-coupled BEC feasible in laboratory.

Results

Cavity-Optomechanics with SO-coupled BEC. Our system consists of a high- Q Fabry-Pérot cavity, with one fixed and one movable mirror, containing SO-coupled BEC illuminated along \hat{y} -axis and coherently driven by single-mode optical field, see Fig. 1(a). Due to high- Q nature of cavity, the intra-cavity photon perform large number of round trips before decaying out, causing strong atom-field coupling. A mechanical oscillator, here movable mirror of cavity, is coupled to the intra-cavity field via radiation pressure force induced by intra-cavity photons. Intra-cavity radiation pressure which will not only excite condensate but also acts as a nonlinear spring that couples the mechanical oscillator with collective density excitation of atomic states¹⁷. To produce SO-coupling, we chose two internal atomic pseudo-spin-states (or laser-dressed hyperfine states

with synthetic half-spin) in $N = 1.8 \times 10^5$ ^{87}Rb bosonic particles having $F = 1$ electronic ground manifold of $5S_{1/2}$ electronic levels and can be labeled in the form of $|\uparrow\rangle = |F = 1, m_F = 0\rangle$ (pseudo-spin-up) and $|\downarrow\rangle = |F = 1, m_F = -1\rangle$ (pseudo-spin-down), as presented in Fig. 1(b). It should be noted that half-spin bosonic particles do not exist in nature, however, they can be engineered with the help of laser in laboratory ³². The magnetic $10G$ bias field or abelian gauge field B_0 is applied along cavity axis (\hat{y} -axis) to induce Zeeman shift $\hbar\omega_z$ between two atomic spin-states $|\uparrow\rangle$ and $|\downarrow\rangle$, where $\omega_z \approx 4.8 \times 2\pi\text{kHz}$. Further, two counter-propagating Raman laser, producing non-abelian gauge field, with wavelength $\lambda = 804.1\text{nm}$ and small detuning $\delta = 1.6E_R$ adjust by bias field B_0 , interacts with atomic spin-states in opposite direction along \hat{x} -axis. The frequencies of these Raman beams are ω_R and $\omega_R + \delta\omega_R$, respectively, with constant frequency difference $\delta\omega_R = \omega_z + \delta/\hbar \simeq 4.8 \times 2\pi\text{MHz}$. and coupling $\Omega_z = 2E_R$. $\mathbf{k}_L = \hbar\mathbf{k}_y = \sqrt{2}\pi\hbar/\lambda$ and $E_R = (\hbar k_y)^2/2m_a = 20 \times 2\pi\text{kHz}$ represents units-less momentum and energy, respectively, where k_y is wave-vector and m_a represents single particle mass.

The external far-off resonant pump field with frequency $\omega_p = \omega_R + \delta\omega_R = 8.8 \times 2\pi\text{MHz}$ and transverse field forms an optical lattice oscillating with frequency $\omega_c = 4 \times 2\pi\text{MHz}$, detuning $\Delta_c = \omega_p - \omega_c = \delta\omega_R$, and decay-rate $\kappa = 0.1\omega_m$. This intra-cavity field, with $\hat{n} = \hat{c}^\dagger\hat{c}$ number of photons, will transfer $2\hat{n}\hbar k$ photon momentum to moving-end mirror by exerting radiation pressure force $F_r = \hat{n}\hbar\omega_c/L = \hat{c}^\dagger\hat{c}\hbar\omega_c/L$ during cavity round trip time $t_r = 2L/c$, where $L = 1.25 \times 10^{-4}$ is the length of cavity in absence of radiation pressure and \hat{c}^\dagger (\hat{c}) are creation (annihilation) operators for intra-cavity optical mode. When mirror frequency $\omega_m = 3.8 \times 2\pi\text{kHz}$ resonants with radiation force $\Delta_c = \pm\omega_m$ (resolved and unresolved-sidebands, respectively), it will excite Stokes

and anti-Stokes scatterings at the boundary of optomechanical system. As, we want to discuss low-temperature dynamics of the system, therefore, the Stokes scattering will be strongly suppressed by operating optomechanical system in resolved-sideband regime $\omega_m \gg \kappa$, which is off-resonant with Stokes scattering so, only anti-Stokes scattering will survive inside the cavity (see Fig. 1(c)). To make our study experimentally possible, we choose these parameters very close to the available experimental studies ^{1, 17, 32, 42}.

The complete Hamiltonian of the system is divided into three parts, $\hat{H} = \hat{H}_a + \hat{H}_m + \hat{H}_f$, where \hat{H}_a accommodates atomic part, \hat{H}_m contains the behavior of mechanical-mode and finally, \hat{H}_f describes intra-cavity field. The strong detuning regime is used to adiabatically eliminate internal excited levels $|e\rangle$ of atomic-mode inside the cavity potential. We further considered rotating frame at external field frequency to obtain effective many-body Hamiltonian for pseudo-spin-1/2 atomic states (H_a), having quantized motion along the cavity axis ^{32, 43, 44}.

$$\hat{H}_a = \int dr \hat{\boldsymbol{\psi}}^\dagger(r) \left(H_0 + V_{LAT} \right) \hat{\boldsymbol{\psi}}(r) + \frac{1}{2} \int dr \sum_{\sigma, \acute{\sigma}} U_{\sigma, \acute{\sigma}} \hat{\psi}_\sigma^\dagger(r) \hat{\psi}_{\acute{\sigma}}^\dagger(r) \hat{\psi}_{\acute{\sigma}}(r) \hat{\psi}_\sigma(r), \quad (1)$$

where m_a is the mass of an atom, $\hat{\boldsymbol{\psi}} = [\hat{\psi}_\uparrow, \hat{\psi}_\downarrow]^\dagger$ represents bosonic field operator for pseudo-spin-up and -down atomic states and $H_0 = \hbar^2 \mathbf{k}^2 \sigma_0 / 2m_a + \tilde{\alpha} \mathbf{k}_x \sigma_y + \frac{\delta}{2} \sigma_y + \frac{\Omega_z}{2} \sigma_z$ describes signal-particle Hamiltonian containing SO-coupling terms ^{29, 70}, where $\tilde{\alpha} = \hbar k_y / 2m$ represents strength of SO-coupling. $\delta = -g\mu_B B_z$ and $\Omega_z = -g\mu_B B_y$ are related to the Zeeman field effects along \hat{z} and \hat{y} axis, respectively, produced by abelian gauge field B_0 . $\sigma_{x,y,z}$ represents 2×2 Pauli matrices under pseudo-spin rotation and σ_0 is the unit matrix ^{32, 70}. Further, $V_{LAT} = \hbar \hat{c}^\dagger \hat{c} U_0 [\cos^2(kx) + \cos^2(ky)]$ accommodates interaction of atomic assemble with intra-cavity field which is in the form of optical lattice coupled to atomic degree of freedom under the assumption $k_x = k_y = k$. Here $\hbar \hat{c}^\dagger \hat{c} U_0$

represents optical potential depth with the effective atom-photon coupling $U_0 = g_0^2/\Delta_a$, where g_0 is the vacuum Rabi frequency, Δ_a is far-off detuning between field frequency and atomic transition frequency ω_0 . Finally, last term explains many-body intra-species and inter-species interactions for atomic spin-states, where $\sigma, \sigma' \in \{\uparrow, \downarrow\}$. $U_{\sigma, \sigma'} = 4\pi a_{\sigma, \sigma'}^2 \hbar^2/m$ accounts for strength of atom-atom interactions, where $a_{\sigma, \sigma'}$ is the s-wave scattering length of atomic-mode.

The Hamiltonian for moving-end mirror is defined as, $\hat{H}_m = \hbar\omega_m \hat{b}^\dagger \hat{b} - i\hbar \frac{g_m}{\sqrt{2}} \hat{c}^\dagger \hat{c}(\hat{b}^\dagger + \hat{b})$, where first term contains information about the motion of mechanical-mode and $\hat{b}(\hat{b}^\dagger)$ are annihilation (creation) operators of moving-end mirror with commutation relation $[\hat{b}^\dagger, \hat{b}] = 1$. Second term accommodates the coupling of moving-end mirror with intra-cavity optical mode caused by the radiation pressure. Here $g_m = \sqrt{2}(\omega_c/L)x_0$ is the coupling strength and $x_0 = \sqrt{\hbar/2m\omega_m}$, is zero point motion of mechanical mirror having mass m . The Hamiltonian representing intra-cavity optical mode is given as, $\hat{H}_c = \hbar\Delta_c \hat{c}^\dagger \hat{c} - i\hbar\eta(\hat{c} - \hat{c}^\dagger)$, where first term describes the strength of intra-cavity optical field and second part is associated with its coupling with external pump field. $|\eta| = \sqrt{P\kappa/\hbar\omega_p}$ defines the coupling of intra-cavity field with input-fields power P .

The eigenenergies of the atomic states are calculated from time-dependent form of Hamiltonian H (see Methods). The strength SO-coupling will create two minima corresponding to lowest energy-levels of atomic spin-states, as can be seen in Fig. 2. In the absence of Raman coupling $\Omega_z = 0\omega_m$, there is no band-gape between lower and upper energy-levels causing the phase mixing of atomic dressed-states, as shown by black curve first panel of Fig. 2. However, in presence of Raman coupling, the band-gape between energy states will appear which increases with increase in

Raman coupling causing quantum phase-transition, from mixed phase to separate phase, of atomic hyperfine states as it varies Zeeman field-effects. It can also be seen that the non-zero Raman detuning $\delta \neq 0\omega_m$ will cause the symmetry-breaking of dispersion over quasi-momentum depending upon the value of detuning. This asymmetric behavior indicates rapid population transfer and enhancement of topological-features emerging in SO-coupled atomic dressed-states.

Fig. 2(d) and Fig. 2(e) show the influence of mechanical-mode coupling with intra-cavity optical mode with SO-coupling $\alpha = 30\pi\omega_m$ and $\alpha = 50\pi\omega_m$, respectively, in the absence of Raman detuning $\delta = 0\omega_m$. The coupling between atomic states and intra-cavity potential is disturbed by the existence of atomic-modes, or vice-versa, when the atomic-modes become resonant with the optical sideband. At this point, atomic-modes will absorb some phonons emitted by the mechanical-mode from intra-cavity field and will behave as a phononic well. Therefore, by increasing the mechanical-mode coupling with intra-cavity field, it will give rise to atomic-state energy levels, as can be seen in Fig. 2(d) and Fig. 2(e), providing control over the topological-properties emerging in SO-coupled atomic dressed-states.

Density-noise spectrum (DNS). To provide realistic understanding of the system, we have to consider the noises affecting the overall dynamics of the device, such as, intra-cavity energy leakage and particularly, noises encountering through Brownian motion of mechanical-mode because of external thermal bath at temperature T . Therefore, we calculate DNS for multiple degree of freedom associated with system. We first find out frequency domain solutions by taking Fourier transform of linearized Langevin equations and solving them for different system compo-

nents. After this, we compute DNS for $\delta q(\omega)$, $\delta q_{\uparrow}(\omega)$ and $\delta q_{\downarrow}(\omega)$, by using standard formalism, $S_{\mathcal{O}} = \frac{1}{2\pi} \int e^{-i(\omega-\dot{\omega})} \langle \mathcal{O}(\omega) \mathcal{O}(\dot{\omega}) \rangle d\dot{\omega}$, where $\mathcal{O}(\omega)$ is generic operator belonging to the system quadratures. By considering the correlation operators of Markovian and Brownian noise in frequency domain, the DNS for pseudo spin- \uparrow and spin- \downarrow atomic states will be read as (see Methods),

$$S_{\uparrow,\downarrow}(\omega, \Delta) = \frac{1}{|X(\omega)|^2} \left(2\pi |C(\omega)|^2 (|B_{\uparrow,\downarrow}(\omega)|^2 + |A_{\downarrow,\uparrow}(\omega)|^2) [\Delta^2 + \kappa^2 + \omega^2] + 2\pi L_{2,4}(\omega) + L_{1,3}(\omega) \frac{\gamma_m \omega}{\omega_m} [1 + \text{Coth}(\frac{\hbar\omega}{2k_B T})] \right), \quad (2)$$

where $X(\omega)$ represents modified susceptibilities for atomic-modes. Fig. 3 illustrates behavior of spin- \uparrow and spin- \downarrow atomic-modes DNS as a function of Δ/ω_m and ω/ω_m . The influence of SO-coupling on atomic back-action of light will not only modify intra-cavity field but also influence the temperature of atomic-modes. Fig. 3(a) demonstrates $S_{\uparrow}(\omega, \Delta)$ in the absence of SO-coupling $\alpha = 0\pi\omega_m$, showing similar behavior as presented in ref. ^{55,58}, however, the presence of inter-species and intra-species interaction will modify DNS. Both the cooling as well as heating mechanisms are observable corresponding to the width and height of $S_{\uparrow}(\omega, \Delta)$, because area under the curve is connected to the temperature of side-mode. One can observe a semi-circular structure appearing with increase in system detuning Δ/ω_m , because of the red-shift in the peak frequency of $S_{\uparrow}(\omega, \Delta)$. Height of the structure initially decreases with increase in detuning Δ/ω_m towards frequency ω/ω_m but shortly, again starts rising with detuning Δ/ω_m . The optimal cooling is achieved at $\Delta = \omega_m/2$ with a considerable shrink in the area underneath atomic DNS. However, in presence of SO-coupling α/ω_m , the height of semi-circular structure appearing in $S_{\uparrow}(\omega, \Delta)$ is suppressed because of the topological-feature emerging in dressed-states modifying intra-cavity energy-levels, as can be seen in Fig. 3(b) and Fig. 3(c), where the strength of SO-coupling is $\alpha = 150\pi\omega_m$ and

$\alpha = 250\pi\omega_m$, respectively. The optimal cooling point is now being shifted to $\Delta = 1\omega_m$. The existence of SO-coupling not only decreases the area underneath $S_{\uparrow}(\omega, \Delta)$ but also suppresses the radius of that semi-circular structure providing control over the low temperature dynamics of atomic states.

Fig. 3(d) shows $S_{\downarrow}(\omega, \Delta)$ verses normalized detuning and frequency in the absence of SO-coupling. At $\alpha = 0\pi\omega_m$, both $S_{\downarrow}(\omega, \Delta)$ and $S_{\uparrow}(\omega, \Delta)$ possess similar behavior because of phase similarity of atomic dressed-states. However, when SO-coupling is non-zero $\alpha \neq 0\pi\omega_m$, $S_{\downarrow}(\omega, \Delta)$ behave differently with $S_{\uparrow}(\omega, \Delta)$, as can be seen in Fig. 3(e) and Fig. 3(f), where the strength of SO-coupling is $\alpha = 150\pi\omega_m$ and $\alpha = 250\pi\omega_m$, respectively. This difference occurs because of topological phase transitions in hyperfine atomic state in presence of SO-coupling. Now the height of semi-circular structure appearing in $S_{\downarrow}(\omega, \Delta)$ is being increased with increase in SO-coupling. However, the radius of semi-circular structure still appears to be decreasing with increase in SO-coupling. Atom-atom interaction U/ω_m of atomic dressed-states show similar influence on $S_{\uparrow}(\omega, \Delta)$ as SO-coupling does to the atomic dressed-states, can be seen in Fig. 3(g-i), where the the strength of atom-atom interactions is considered as, $U = 5.5\omega_m, 7.5\omega_m, 9.5\omega_m$, respectively. The radius as well as height of to atomic DNS decreases with increase in atom-atom interactions U/ω_m of dressed-states. (Note: The effects of atom-atom interactions $S_{\downarrow}(\omega, \Delta)$ are not shown here because they will be like-wise as on $S_{\uparrow}(\omega, \Delta)$.) The strength of atom-atom interactions can likely be used to control the low-temperature dynamics of atomic dressed-states as SO-coupling.

Similarly, dynamics of mechanical-mode will be influenced by the existence of atomic-sates

and its topological-features as atomic dressed-states are influenced by the existence of mechanical-mode. To analyze these effects, we calculated DNS for mechanical-mode by following similar process as for DNS of atomic-modes in the form of two frequency auto-correlation,

$$S_m(\omega, \Delta) = \frac{1}{|X_m(\omega)|^2} \left(|A_m(\omega)|^2 (\Delta^2 + \kappa^2 + \omega^2) + 2\pi B_m(\omega) + C_m(\omega) \frac{\gamma_m \omega}{\omega_m} [1 + \text{Coth}(\frac{\hbar \omega}{2k_B T})] \right), \quad (3)$$

where $X_m(\omega)$ corresponds to the modified susceptibility of mechanical-mode. Fig. 4 demonstrates $S_m(\omega, \Delta)$ as a function of Δ/ω_m and ω/ω_m , under the influence of SO-coupling and atom-atom interaction. The atom-field coupling is considered as $G_a = 4.1\omega_m$ while the mirror-field coupling is taken as $G_m = 1.5\omega_m$. The behavior of $S_m(\omega, \Delta)$ in the absence of SO-coupling, causing zero non-abelian field, is shown in Fig. 4(a), which is similar to behavior of atomic DNS. A semi-circular structure appears with increase in detuning Δ/ω_m towards frequency ω/ω_m . The height of $S_m(\omega, \Delta)$ decreases initially and achieves optimal cooling point. However, when the system detuning is further increased from $\Delta = 1\omega_m$, the $S_m(\omega, \Delta)$ shows rapid increase in the height of structure giving rise to the temperature of mechanical-mode.

The strength of SO-coupling induces similar influence as it does to the atomic DNS. The radius of the structure is suppressed by the increase SO-coupling, as shown in Fig. 4(b) and Fig. 4(c), where the strength of SO-coupling is increased to $\alpha = 60\pi\omega_m$ and $\alpha = 120\pi\omega_m$, respectively. Not only SO-coupling but also the atom-atom interactions of atomic dressed-states will show similar effects on mechanical DNS as they are inducing in atomic DNS. The increase in atom-atom interactions will also reduce the radius of semi-circular structure, as shown in Fig. 4(d), Fig. 4(e) and Fig.

4(f), where the strength of atom-atom interactions is increased to $U = 5.5\omega_m, 7.5\omega_m, 9.5\omega_m$, respectively. The SO-coupling and atom-atom interactions modifies the atomic density mode excitation leading to the variation in intra-cavity optical spectrum. As the atomic-modes and mechanical-mode are connected with each other through intra-cavity radiation pressure, acting as a spring between these two independent entities, therefore, the modifications in intra-cavity potential, through SO-coupling or atom-atom interaction, will show similar influence on mechanical-mode as they are producing in atomic dressed-states.

Effective-temperature of mechanical-mode. To further enhance our understanding, we calculate effective-temperature of mechanical-mode and analyze its behavior under the influence of SO-coupling and atom-atom interactions. The effective-temperature of mechanical mode (T_{eff}) is calculated by formula $T_{eff} = \langle E_m \rangle / k_B$, where $\langle E_m \rangle = m\omega_m^2 \langle \delta \hat{q}^2 \rangle / 2 + \langle \delta \hat{p}^2 \rangle / 2m$, corresponds to the mean energy $\langle E_m \rangle$ which is obtained by measuring area under $S_m(\omega, \Delta)$. The position and momentum variances are, $\langle \delta \hat{q}^2 \rangle = \frac{1}{2\pi} \int S_m(\omega, \Delta) d\omega$ and $\langle \delta \hat{p}^2 \rangle = m^2 \omega_m^2 S_m(\omega, \Delta)$, where m is the mass of mechanical-mode. Fig. 5(a) represents T_{eff} of mechanical-mode under the influence of SO-coupling at frequency $\omega = 0.1\omega_m$. The atom-field and mirror-field coupling are taken as, $G_m = 1\omega_m$ and $G_a = 25\omega_m$, respectively and the atom-atom interactions strength is $U = 5.5\omega_m$. The Raman coupling is considered as, $\Omega_z = \omega_m$ while the Raman laser detuning is also fixed to $\delta = \omega_m$. The blue curve illustrates temperature profile at $\alpha = 0\pi\omega_m$, where the lowest temperature is achieved at $\Delta \approx 1.3\omega_m$. The red, green and orange represents T_{eff} of mechanical-mode when the SO-coupling is increased to $\alpha = 70\pi\omega_m$, $\alpha = 100\pi\omega_m$ and $\alpha = 130\pi\omega_m$, respectively. The existence of SO-coupling suppresses the interval of optimal temperature over system

detuning Δ/ω_m due to cavity mode excitation which transfers energy to mechanical-mode via radiation pressure. However, if we increase the atomic-mode coupling with intra-cavity field, atomic dressed-states will absorb more phonons emitted by mechanical-mode of the system which will decrease the thermal excitation of mechanical-mode. Fig. 5(b) shows such influence of atom-field coupling on mechanical oscillator of the system where the effective-temperature of mirror is decreased by increasing atom-field coupling. The atom-atom interactions of atomic-mode will influence the mechanical-mode similarly as SO-coupling have shown. The atom-atom interaction also contributes to increase the thermal excitation of mechanical-mode as shown in Fig. 5(c).

Further, to understand the influence of topological-features produced by SO-coupled atomic dressed-states on system mechanical oscillator, we plotted effective-temperature of mechanical-mode, as a function of atom-atom interactions U/ω_m and normalized SO-coupling α/ω_m , in absence (Fig. 5(d)) and in presence (Fig. 5(e)) of Raman coupling Ω_z/ω_m . Fig. 5(d) shows that the maximum effective-temperature of mechanical-mode T_{eff} , in absence of Raman coupling, is centered at atom-atom interactions $U \approx 6\omega_m$ and it appeared to be saturated with increase in SO-coupling. One can state that the maxima of mechanical-mode effective-temperature shows linear and localized behavior with respect to atom-atom interactions and SO-coupling. On the other hand, in presence of Raman coupling the maxima of effective-temperature show exponential increase with atom-atom interactions and SO-coupling, as can be seen in Fig. 5(e), where the strength of Raman field coupling is increased to $\Omega_z = 60\omega_m$. The increase in Raman coupling will enhance Zeeman field effects which produces band-gape between energy eigen-states of atomic-mode. These topological-features will modify intra-cavity phononic relation between

atomic dressed-states and mechanical oscillator. The nonlinear behavior of maximum mechanical-mode heating is caused by these topological-effects and can be further enhanced by increasing Zeeman field effects providing an opportunity to control low-temperature dynamics of optomechanical devices with the help of these topological-features.

Dynamic structure factor (DSF). The phenomenon of DSF ($S_D(k, \omega)$) has a significant importance to determine the density fluctuations, depending upon the inter-particle correlations and time evolution, and their influences on many-body systems. The DSF is usually computed through the Fourier transform of spatial or temporal density-density auto-correlation. In our case, we are utilizing auto-correlation of quantum fluctuation operators in Fourier domain to analyze low-temperature dynamics of the system. Therefore, we compute DSF by manipulating dynamical fluctuation operators. Here we analyze DSF, at one particular wave-vector k , by computing the light leaking-out of the cavity which reveals the information about Fourier domain auto-correlations of multiple degrees of freedom related to system. The output field also describes the evolution of intra-cavity quasi-particle excitations from atomic-mode through intra-cavity synthetic gauge field. The resultant DSF of the system, as a function of intra-cavity wave-vector k and frequency ω , can be related to spectral density of out-going mode as ⁶⁹,

$$S_D(k, \omega) = \frac{4(\kappa^2 + \Delta^2)}{N\eta^2} \left(\frac{1}{2\pi} S_{out}(P, \omega) + n_s^2 \delta(\omega) \right), \quad (4)$$

where n_s is the steady-state photon number of intra-cavity optical field and $S_{out}(P, \omega)$ is DNS of out-going optical mode containing information about spectral densities corresponding to fluctuation operators of atomic dressed-states $S_{\uparrow, \downarrow}(\omega, \Delta)$ and mechanical-mode $S_m(\omega, \Delta)$ (see methods). The frequency ω is referred to the shifted frequency of external input field after interacting with

system which cause inelastic scattering of photons.

Fig. 6 demonstrates $S_{out}(P, \omega)$, as a function of normalized frequency ω/ω_m and normalized input field power P/P_{cr} , for both in presence as well as in absence of SO-coupling cases and shows $S_D(k, \omega)$ based on cavity output field for different input powers. Fig. 6(a) and (b) illustrate DNS of out-going optical mode in absence and in presence of SO-coupling, respectively. The strengths of atom-atom interactions, Zeeman field effects and Raman detuning are considered as $U = 5.5\omega_m$, $\Omega_z = 1\omega_m$ and $\delta = \omega_m$, respectively. The out-going optical field contains inelastically scattered input pump field from intra-cavity atomic dressed-states and mechanical-mode. Fig. 6(a) demonstrates $S_{out}(P, \omega)$, in the absence of SO-coupling, where we can observe two sidebands, referred to resolved and unresolved-sideband, appearing at $\omega < 0\omega_m$ and $\omega > 0\omega_m$, caused by the incoherent creation and annihilation of quasi-particles⁶⁹, respectively. If we increase the input power, the both sidebands tends to move towards $\omega = 0\omega_m$ because of the decrease in the spectral strength of quasi-particles due to quantum fluctuations of the system, as can be seen in Fig. 6(a). It is, intuitively, referred to the scattering of intra-cavity optical mode at Bragg planes in the density-modulated cloud. The both sidebands seem to get mixed with each other due to the presence of another structure approximately at $P \approx 6P_{cr}$. This point is referred to critical point and the secondary structure, which is centered at $\omega = 0\omega_m$, is caused by quantum noise associated with system⁶⁹. However, in presence of SO-coupling, this critical point is shifted towards input field power, with new value $P \approx 7P_{cr}$, due to the modifications in the inelastic scattering of intra-cavity field by atomic spin-phase transitions⁶⁵, as shown in Fig. 6(b), where the the SO-coupling is $\alpha = 80\pi\omega_m$. It can also be noted that the spectral strength of both sidebands as well as secondary

structure is increased by the existence of SO-coupling. Precisely saying, the SO-coupling will cause the addition of quasi-particles excited from atomic dressed-states in presence of non-abelian gauge field. These quasi-particles are referred to anyons having neither bosonic nor fermions nature. These non-abelian anyons possess great interest in topological quantum computation because of their implications as quantum-gates.

The second panel in Fig. 6 contain DSF, obtained from equation (4), for different input pump field powers corresponding to $S_{out}(P, \omega)$ shown in first panel. Fig. 6(c) illustrates $S_D(k, \omega)$ at power ratio $P/P_{cr} = 3$ and shows emergence of two sidebands at $\omega \approx -3.5\omega_m$ and $\omega \approx 3\omega_m$ corresponding to creation and annihilation of quasi-particles, respectively, caused by fluctuating dynamics of the system as shown in first panel. Another, comparatively small, fluctuating structures can be seen at $\omega = 0\omega_m$ which is induced by the quantum-noise effects. These structures amazingly verify the experimental finding of reference ⁶⁹. On the contrary, if we increase the power of the system, the strength of DSF will be suppressed by the enhancement in system fluctuations and the peaks of sidebands will move towards $\omega = 0\omega_m$, as can be seen in Fig. 6(d), where the input power is $P/P_{cr} = 5$. However, the structure appearing at $\omega = 0\omega_m$ is enhanced due to the increase in quantum noises. DSF also describes mean occupation number which is basically related to the temperature of multiple components of system like mechanical degree of freedom and atomic spin-states. Therefore, DSF can help us in extracting information about heating or cooling mechanisms of the optomechanical system. Further, the presence of SO-coupling will also modify DSF because of non-abelian quasi-particle excitations, as can be seen in Fig. 6(e) and (f), possessing similar behavior corresponding to out-going optical mode shown in Fig. 6(b). The strength

of sideband peaks appearing in $S_D(k, \omega)$ are increased due to the addition of non-abelian quasi-particle. However, the secondary structure appearing because of quantum noises is now being suppressed due to the existence of SO-coupling.

In order to get better understanding of influences produced by SO-coupling, generating topological-effects and cavity mediated long-range atomic interaction on DSF, we plot $S_D(k, \omega)$ for multiple values of SO-coupling and atom-atom interactions. Fig. 7(a) carries $S_D(k, \omega)$ as a function of normalized frequency for difference strengths of SO-coupling at input field power $P = 5P_{cr}$. It is clear that by increasing the SO-coupling strength sidebands of quasi-particle modes are enhanced and shifted away from $\omega = 0\omega_m$ due to the addition of non-abelian quasi-particles. However, quantum-noise fluctuations, appearing at $\omega \approx 0$, are now being suppressed by increasing SO-coupling causing enhancement in intra-cavity atomic back-action of light. This significant feature will enhance the utilization of SO-coupling induced non-abelian gauge field in complex system. Besides, if we increase many-body interactions in atomic spin-states, it will show further alterations in $S_D(k, \omega)$, as can be seen in Fig. 7(b), where the SO-coupling is fixed to $\alpha = 60\pi\omega_m$ and other couplings are kept same as in the previous discussions. The increase in the strength of atom-atom interactions are showing similar effects not only on sidebands in the form of enhancement in strength of quasi-particles modes but also on the secondary peak appearing in $S_D(k, \omega)$. These modifications are emerging from influence of SO-coupled and interacting atomic dressed-states on atomic back-action light inside the cavity which will modify inelastic photon scattering.

Discussion

We have demonstrated cavity-optomechanics with SO-coupled Bose-Einstein condensate (BEC) by using two Raman beams producing non-abelian synthetic gauge field. The set-up provides a flexible and realistic platform to observe the influences of SO-coupling induced topological-features and atom-atom interactions on dynamics of both atomic-modes as well as mechanical-mode in presence quantum-noises by calculating density-noise spectrum (DNS), which is directly connected to the low-temperature profile, and dynamic structure factor (DSF), which deals with quasi-particle excitations. The numerical results show us that not only SO-coupling and atom-atom interactions provide effective control over the low-temperature dynamics, of both atomic dressed-states as well as mechanical-mode, but also enable us to optomechanically control the topological- features emerging in system via synthetic gauge field. The coupling of mechanical oscillator with intra-cavity field transfers phonons to the atomic dressed-states increasing atomic excitation energy and cause the modifications in topological nature of the system. On the other hand, these topological-effects are nonlinearly affecting low-temperature profile of mechanical oscillator. Thus, we can control the low-temperature dynamics of atomic and mechanical oscillators by maneuvering applied SO-coupling strength and long-range atomic interactions.

Moreover, we calculate DSF by manipulating DNS of out-going optical mode to understand the quasi-particles excitations ⁶⁹. The strengths of SO-coupling and atom-atom interactions are not only controlling the out-going optical field but also enables us to manage DSF describing quasi-particle modes. These features provide us stunning opportunity to analyze anyons quasi-

particles excited by non-abelian gauge field. It has also been found that the DSF contains another fluctuating structures centered at $\omega \approx 0\omega_m$, depending upon system power, corresponding to the presence of quantum-noise which is significant according to the experimental finding in reference ⁶⁹. However, these noise effects can be suppressed by increasing the strength of SO-coupling and atom-atom interaction. From practical point of view, we have provided a particular set of parameters and procedures very close to the present experimental ventures. Hence our study of cavity-optomechanics with SO-coupled BEC is experimentally feasible in laboratory. Our findings not only provide an applicable platform to investigate optomechanical behavior of SO-coupling induced topological-features, like spin-Hall effects and topological-insulators, but also enhance the utilization of interacting and SO-coupled BEC in the field of quantum optics and quantum information, which would be a milestone towards topological quantum computation.

Methods

Atomic eigenenergies calculation. Here we provide some details about dispersion calculation of atomic states. We substitute plane-wave ansatz $\hat{\psi}(r) = e^{ikr}\hat{\varphi}$, where $\hat{\varphi} = [\hat{\varphi}_\uparrow, \hat{\varphi}_\downarrow]^\dagger$ in equation 1, by considering homogeneous atomic-modes distribution with normalization condition $|\hat{\varphi}_\uparrow|^2 + |\hat{\varphi}_\downarrow|^2 = N$. We assume that the strengths of intra-species interactions of both spin-states are equal with each other and are defined as, $U_{\uparrow,\uparrow} = U_{\downarrow,\downarrow} = U$. Similarly, inter-species interactions can be modeled as, $U_{\uparrow,\downarrow} = U_{\downarrow,\uparrow} = \varepsilon U$, where parameter ε depends upon the incident laser configuration ³². Under these considerations, we solve equation 1 and compute quantum Langevin equation. To analyze the dynamics of system in a realistic way, we consider the effects of dissipation and noises

associated with optomechanical system which is possible with the help of standard quantum-noise operators ^{55,56}. The quantum Langevin equation concept leads us to develop coupled and time dependent set of equations, containing noise operators for optical, mechanical and atomic degrees of freedom,

$$\frac{d\hat{c}}{dt} = \dot{\hat{c}} = (i\tilde{\Delta} + i\frac{g_m}{\sqrt{2}}(\hat{b} + \hat{b}^\dagger) - ig_a\hat{\phi}^\dagger\hat{\phi} - \kappa)\hat{c} + \eta + \sqrt{2\kappa}a_{in}, \quad (5)$$

$$\frac{d\hat{b}}{dt} = \dot{\hat{b}} = -\omega_m\hat{b} - \frac{g_m}{\sqrt{2}}\hat{c}^\dagger\hat{c} - \gamma_m\hat{b} + \sqrt{\gamma_m}f_m, \quad (6)$$

$$\begin{aligned} \frac{d\hat{\phi}}{dt} = \dot{\hat{\phi}} = & \left(\frac{\hbar\mathbf{k}^2\sigma_0}{2m} + \tilde{\alpha}\mathbf{k}_x\sigma_y + \frac{\delta}{2}\sigma_y + \frac{\Omega}{2}\sigma_z - \gamma_a + g_a\hat{c}^\dagger\hat{c} \right)\hat{\phi} + \frac{1}{2}U\hat{\phi}^\dagger\hat{\phi}\hat{\phi} \\ & + \frac{1}{2}\varepsilon U\hat{\phi}_\sigma^\dagger\hat{\phi}_\sigma\hat{\phi}_\sigma + \sqrt{\gamma_a}f_a, \end{aligned} \quad (7)$$

where $\tilde{\Delta} = \Delta_c - NU_0/2$ is the effective detuning of the system and \hat{c}_{in} is Markovian input noise operator, having zero-average $\langle \hat{c}_{in}(t) \rangle = 0$ and delta-correlation $\langle \hat{c}_{in}(t)\hat{c}_{in}(\hat{t}) \rangle = \delta(t - \hat{t})$ under the condition $\hbar\omega_c \gg k_B T$, associated with intra-cavity field and induced by the leaky cavity mirror. The term γ_m describes mechanical energy decay rate of the moving-end mirror and \hat{f}_m is noise operator (or zero-mean Langevin-force operator) connected with the Brownian motion of mechanical-mode of the system and can be defined by using non-Markovian correlation ^{55,56} $\langle \hat{f}_m(t)\hat{f}_m(\hat{t}) \rangle = \frac{\gamma_m}{2\pi\omega_m} \int d\omega e^{-i\omega(t-\hat{t})} [1 + Coth(\frac{\hbar\omega}{2k_B T})]$. The external harmonic trapping potential of the condensate, which we have ignored so far because it appeared to be spin independent, couples the atomic states to modes associated with system, resulting in the damping of atomic motion. The parameter γ_a represents damping of atomic dressed-states motion while \hat{f}_a is the associated noise operators assumed to be Markovian with delta-correlation $\langle \hat{f}_a(t)\hat{f}_a(\hat{t}) \rangle = \delta(t - \hat{t})$ under the condition $\hbar\Omega \gg k_B T$. Further, $\sigma, \hat{\sigma} \in \{\uparrow, \downarrow\}$ and $g_a = \frac{\omega_c}{L} \sqrt{\hbar/m_{bec}4\omega_r}$ is the coupling of atomic-mode with intra-cavity field, having effective mass of BEC $m_{bec} = \hbar\omega_c^2/(L^2U_0^2\omega_r)$.

Further, by adopting a mean-field approximation, we consider the intra-cavity field steady-state and replace the intra-cavity field operator by its expectation value $\hat{c} \rightarrow \langle c \rangle \equiv c_s$. To calculate energy dispersion E_N of atomic-modes, we define E_N as the solution of time-independent Langevin equations and replace the time derivative id/dt with eigenenergy E_N . After doing some mathematics and utilizing Pauli matrices, the coupled Langevin equations will take the form,

$$n_s = c_s^\dagger c_s = \frac{\eta}{\kappa^2 + (\tilde{\Delta} - \frac{g_m}{\sqrt{2}}(\hat{b}^\dagger + \hat{b}) + g_a(\hat{\varphi}_\uparrow^\dagger \hat{\varphi}_\uparrow + \hat{\varphi}_\downarrow^\dagger \hat{\varphi}_\downarrow))^2}, \quad (8)$$

$$\hat{b} = \frac{g_m c_s^\dagger c_s}{\sqrt{2}(E_N + i\omega_m + \gamma_m)}, \quad \hat{b}^\dagger = \frac{g_m c_s^\dagger c_s}{\sqrt{2}(E_N - i\omega_m + \gamma_m)}, \quad (9)$$

$$E_N \begin{pmatrix} \hat{\varphi}_\uparrow \\ \hat{\varphi}_\downarrow \end{pmatrix} = \begin{pmatrix} \frac{\hbar \mathbf{k}^2}{2m} + \frac{\Omega_z}{2} + g_a c_s^\dagger c_s + \frac{1}{2}UN - \gamma_a & -i(\alpha \mathbf{k}_x + \frac{\delta}{2}) + \frac{1}{2}U(\varepsilon - 1)\hat{\varphi}_\downarrow^\dagger \hat{\varphi}_\uparrow \\ i(\alpha \mathbf{k}_x + \frac{\delta}{2}) + \frac{1}{2}U(\varepsilon - 1)\hat{\varphi}_\uparrow^\dagger \hat{\varphi}_\downarrow & \frac{\hbar \mathbf{k}^2}{2m} - \frac{\Omega_z}{2} + g_a c_s^\dagger c_s + \frac{1}{2}UN - \gamma_a \end{pmatrix} \begin{pmatrix} \hat{\varphi}_\uparrow \\ \hat{\varphi}_\downarrow \end{pmatrix} \quad (10)$$

where n_s is the steady-state photon number inside cavity. For simplicity, we have ignored quantum noises associated with system while calculating eigenenergies of atomic-mode. By solving equation 10, for $\hat{\varphi}_\uparrow$ and $\hat{\varphi}_\downarrow$, and assuming eigenenergies of moving-end mirror and atomic-mode independent, we plot the roots of eigenenergies E_N versus quasi-momentum \mathbf{k}_x , as shown in Fig. 2. (Note that we consider $\mathbf{k}_y = \mathbf{k}_z = 0$ because SO-coupling occurs only in the direction of \hat{x} -axis.)

Langevin equations and frequency domain solutions. The coupled Langevin equations of the system contain nonlinear terms in the form of coupling among different degrees of freedom and noises associated with system. By considering intense external pump field, these equations can be linearized with the help of quantum fluctuations as, $\hat{\mathcal{O}}(t) = \mathcal{O}_s + \delta\mathcal{O}(t)$, where \mathcal{O} can be any operator of the system, \mathcal{O}_s represents steady-state value and $\delta\mathcal{O}(t)$ is the first order quan-

tum fluctuation. During these calculation for simplicity, we assume that the both atomic states, spin- \uparrow and spin- \downarrow , have equal amount of particles, i.e $\hat{\varphi}_\uparrow^\dagger \hat{\varphi}_\uparrow = \hat{\varphi}_\downarrow^\dagger \hat{\varphi}_\downarrow = N/2$. Further more, we define system quadratures in the form of dimensionless position and momentum quantum quadratures as, $\hat{q}_O = \frac{1}{\sqrt{2}}(\hat{O} + \hat{O}^\dagger)$ and $\hat{p}_O = \frac{i}{\sqrt{2}}(\hat{O} - \hat{O}^\dagger)$, respectively, (O is generic operator) having commutation relation $[\hat{q}_O, \hat{p}_O] = i$ which reveals the value of scaled Planck's constant $\hbar = 1$. Now the linearized Langevin equation are defined in form of $\dot{\mathcal{X}} = \mathcal{K}\mathcal{X} + \mathcal{F}$, where vector $\mathcal{X} = [\delta q_c(t), \delta p_c(t), \delta q_m(t), \delta p_m(t), \delta q_\uparrow(t), \delta p_\uparrow(t), \delta q_\downarrow(t), \delta p_\downarrow(t)]^\tau$ contains position and momentum quadratures of the system and vector $\mathcal{F} = [\sqrt{2\kappa}q_c^{in}, \sqrt{2\kappa}p_c^{in}, 0, 2\sqrt{\gamma_m}f_m, 0, 2\sqrt{\gamma_a}f_a, 0, 2\sqrt{\gamma_a}f_a]^\tau$ defines noises and damping associated with system. The matrix \mathcal{K} contains dynamical parameters associated with system,

$$\mathcal{K} = \begin{pmatrix} -\kappa & \Delta & 0 & 0 & 0 & 0 & 0 & 0 \\ \Delta & -\kappa & -G_m & 0 & G_a & 0 & G_a & 0 \\ -2G_m & 0 & -\gamma_m & \omega_m & 0 & 0 & 0 & 0 \\ 0 & 0 & -\omega_m & -\gamma_m & 0 & 0 & 0 & 0 \\ 2G_a & 0 & 0 & 0 & M & \frac{\Omega_z}{2} & (\alpha - \frac{\delta}{2}) & 0 \\ 0 & 0 & 0 & 0 & \frac{\Omega_z}{2} & M & 0 & -(\alpha - \frac{\delta}{2}) \\ 2G_a & 0 & 0 & 0 & (-\alpha + \frac{\delta}{2}) & 0 & M & -\frac{\Omega_z}{2} \\ 0 & 0 & 0 & 0 & 0 & -(-\alpha + \frac{\delta}{2}) & -\frac{\Omega_z}{2} & M \end{pmatrix},$$

where $M = \frac{\Omega}{2} + v + UN(1 - \varepsilon) - \gamma_a$, $v = g_a n_s$ and $\Omega = \hbar \mathbf{k}^2 / m_a$ is the recoil frequency of atomic states. $\alpha = \tilde{\alpha} \mathbf{k}_x$ is the effective strength of SO-coupling. The evolution of the system can be analyzed by matrix \mathcal{K} which contains multiple crucial parameters such as effective detuning of the system $\Delta = \tilde{\Delta} - g_m q_s + g_a N$, where q_s is steady-state quadratures of mechanical-mode, and modified coupling of intra-cavity optical mode with mechanical $G_m = \sqrt{2}g_m |c_s|$ mode and atomic-modes $G_a = \sqrt{2}g_a |c_s|$, tuned by the mean intra-cavity optical mode amplitude $c_s = \frac{\eta}{\kappa + i\Delta}$.

The particular interlaced nature of these steady-state parameters provides an efficient opportunity to understand nonlinear and bistable dynamics of the system.

To make the system accurate and useful, we have to ensure stability of the system and for this purpose, we perform stability analysis of the system. The system can only be stable if the roots of the characteristic polynomial of matrix \mathcal{K} lie in the left half of the complex plane. For this purpose, we apply Routh-Hurwitz Stability Criterion ⁵⁸ on matrix \mathcal{K} and numerically developed stability conditions for the system. These stability conditions are given as, $M > \kappa + \gamma_m$, $(\alpha - \delta/2)^2 + M^2 > \kappa^2 + \Delta^2 - \omega_m^2 - \Omega_z^2$, $\omega_m > \Delta > \kappa > \gamma_m > 0$ and finally, $\Delta G_a^2 + \Delta G_m^2 > M(\kappa^2 - \Omega_z^2)$. We strictly follow these conditions while performing all numerical calculations in the manuscript.

Furthermore, we take Fourier transform of linearized Langevin equations to perform frequency domain analysis and solve them for position and momentum quadratures of intra-cavity field,

$$\begin{aligned} \delta q_c(\omega) = & \frac{1}{L(\omega)} \left(\sqrt{2\kappa} [\Delta \delta p_c^{in} + (\kappa + i\omega) \delta q_c^{in}] \right. \\ & \left. + \Delta [G_a \delta q_{\uparrow}(\omega) + G_a \delta q_{\downarrow}(\omega) - G_m \delta q(\omega)] \right), \end{aligned} \quad (11)$$

$$\begin{aligned} \delta p_c(\omega) = & \frac{1}{L(\omega)} \left(\sqrt{2\kappa} [\Delta \delta q_c^{in} + (\kappa + i\omega) \delta p_c^{in}] \right. \\ & \left. + (\kappa + i\omega) [G_a \delta q_{\uparrow}(\omega) + G_a \delta q_{\downarrow}(\omega) - G_m \delta q(\omega)] \right), \end{aligned} \quad (12)$$

respectively, position quadrature of atomic-modes,

$$\begin{aligned} \delta q_{\uparrow,\downarrow}(\omega) = & \frac{1}{X(\omega)} \left((B_{\uparrow,\downarrow}(\omega) + A_{\downarrow,\uparrow}(\omega)) C(\omega) [\Delta \delta p_c^{in} \right. \\ & \left. + (\kappa + i\omega) \delta q_c^{in}] + L_{1,3}(\omega) f_m + L_{2,4}(\omega) f_a \right), \end{aligned} \quad (13)$$

and finally for the position quadrature of mechanical-mode,

$$\delta q_m(\omega) = \frac{1}{X_m(\omega)} \left(A_m(\omega) [\Delta \delta p_c^{in} + (\kappa + i\omega) \delta q_c^{in}] + B_m(\omega) f_m + C_m(\omega) f_a \right), \quad (14)$$

where $L(\omega) = (\kappa + i\omega)^2 - \Delta^2$ contains detuning of the system, $W(\omega) = \gamma_a + i\omega - \Omega/2 - v - UN(1 - \varepsilon)$, $K(\omega) = W^2(\omega) + (\alpha^2 - \delta/2)^2$ describes atom-atom interactions and $S_m(\omega) = (\gamma_m + i\omega)^2 L(\omega) - L(\omega) \omega_m^2 + 2G_m^2 \Delta (\gamma_m + i\omega)$ is related to mirror coupling with intra-cavity field. $A_{\uparrow, \downarrow}(\omega) = 4W(\omega)K(\omega)L(\omega)S_m(\omega) \pm \Omega_z^2 L(\omega)S_m(\omega) - 8G_a^2 \Delta K(\omega)S_m(\omega) + 16G_a^2 \Delta^2 G_m^2 (\gamma_m + i\omega)K(\omega)$ and $B_{\uparrow, \downarrow}(\omega) = \pm \Omega_z^2 L(\omega)S_m(\omega) + 4(\pm \alpha \mp \delta/2)K(\omega)L(\omega)S_m(\omega) + 8G_a^2 \Delta K(\omega)S_m(\omega) - 16G_a^2 \Delta^2 G_m^2 (\gamma_m + i\omega)K(\omega)$ describes the behavior of atomic-mode and its association with moving-end mirror of the system. $B_m(\omega) = 2G_m \sqrt{(2\kappa)} (\gamma_m + i\omega) X(\omega) + 2G_m \Delta G_a (A_{\uparrow}(\omega) + A_{\downarrow}(\omega) + B_{\uparrow}(\omega) + B_{\downarrow}(\omega))$ represents mechanical-mode behavior and its coupling with atomic-modes. Further, $C(\omega) = 8G_a^2 \sqrt{(2\kappa)} K(\omega)S_m(\omega) + 16G_a^2 \sqrt{2\kappa} G_m^2 (\gamma_m + i\omega)K(\omega)$, $L_{1,3}(\omega) = (B_{\uparrow, \downarrow}(\omega) + A_{\downarrow, \uparrow}(\omega))8G_a^2 \sqrt{\gamma_m} \Delta K(\omega)L(\omega)(\gamma_m + i\omega)$, $L_{2,4}(\omega) = (B_{\uparrow, \downarrow}(\omega) + A_{\downarrow, \uparrow}(\omega))8G_a^2 \sqrt{\gamma_a} K(\omega)S_m(\omega)$, $B_m(\omega) = 2G_m \Delta G_a (L_1(\omega) + L_3(\omega)) + 2\sqrt{\gamma_m} L(\omega)X(\omega)$ and $C_m(\omega) = 2G_m \Delta G_a (L_2(\omega) + L_4(\omega))$. The term $X(\omega) = A_{\uparrow}(\omega)A_{\downarrow}(\omega) + B_{\uparrow}(\omega)B_{\downarrow}(\omega)$ represents modified susceptibility atomic states and $X_m(\omega) = X(\omega)S_m(\omega)$ corresponds to the modified susceptibility of mechanical-mode.

Spectral density of out-going optical mode. In order to calculate output optical mode of the system, we use input-output field relation as, $\delta q_c^{out} = \sqrt{2k} \delta q_c - \delta q_c^{in}$ and $\delta p_c^{out} = \sqrt{2k} \delta p_c - \delta p_c^{in}$, where p_{in}, q_{in} and p_{out}, q_{out} represent input and output field quadratures, respectively. By utilizing

above relation and intra-cavity field quadrature, we obtain output field relation as,

$$\delta q_c^{out}(\omega) = \frac{1}{L(\omega)} \left([2\kappa\Delta\delta p_c^{in} + (\kappa^2 + \omega^2 + \Delta^2)\delta q_c^{in}] + \sqrt{2\kappa}\Delta [G_a\delta q_\uparrow(\omega) + G_a\delta q_\downarrow(\omega) - G_m\delta q(\omega)] \right), \quad (15)$$

$$\delta p_c^{out}(\omega) = \frac{1}{L(\omega)} \left([2\kappa\Delta\delta q_c^{in} + (\kappa^2 + \omega^2 + \Delta^2)\delta p_c^{in}] + \sqrt{2\kappa}(\kappa + i\omega) [G_a\delta q_\uparrow(\omega) + G_a\delta q_\downarrow(\omega) - G_m\delta q(\omega)] \right). \quad (16)$$

By combining position and momentum quadratures of field, we obtain out-going field operator c_{out} given as,

$$\delta c_{out}(\omega) = \frac{1}{L(\omega)} \left([2\kappa\Delta\delta c_{in}^\dagger + (\kappa^2 + \omega^2 + \Delta^2)\delta c_{in}] + \sqrt{2\kappa}\Delta [G_a\delta q_\uparrow(\omega) + G_a\delta q_\downarrow(\omega) - G_m\delta q(\omega)] \right). \quad (17)$$

Further, to determine the dependence of out-going optical mode on the external pump field power P , we redefine coupling terms as a function of P ,

$$G_m = \sqrt{2}C_S g_m = \frac{2\omega_c}{L} \sqrt{\frac{P\kappa}{m\omega_m\omega_p(\kappa^2 + \Delta^2)^2}}, \quad (18)$$

$$G_a = \sqrt{2}C_S g_a = \frac{2\omega_c}{L} \sqrt{\frac{P\kappa}{m\Omega\omega_p(\kappa^2 + \Delta^2)^2}}, \quad (19)$$

and after this, we calculate DNS of out-going optical mode by simply using two frequency auto-correlation formula (as discussed in main text),

$$S_{out}(P, \omega) = \frac{2\pi}{|L(\omega)|^2} \left([\kappa^2 + \omega^2 + \Delta^2 + 2\kappa\Delta] + 4\kappa\Delta [G_a S_\uparrow(\omega, \Delta) + G_a S_\downarrow(\omega, \Delta) - G_m \delta S_m(\omega, \Delta)] \right). \quad (20)$$

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Figure 1 | Cavity-Optomechanics with spin-orbit coupled Bose-Einstein condensate.

(a) Schematic diagram of spin-orbit (SO)-coupled ^{87}Rb Bose-Einstein condensate (BEC) trapped inside optomechanical cavity whose length L depends on the mechanical oscillator's displacement q , here one of the cavity mirrors moving with frequency ω_m . External pump field with frequency ω_p , along \hat{y} -axis, will generate intra-cavity optical mode having frequency ω_c . The mechanical oscillator is coupled to the intra-cavity field via radiation pressure force induced by intra-cavity photons. Due to intra-cavity photon recoil, matter-wave momentum modes will be excited that modulates the atomic states density distribution. These modulation behave formally like a mechanical oscillator that can shift the cavity resonance frequency. A bias abelian magnetic field B_0 is applied to intra-cavity atomic-mode to generate Zeeman shift $\hbar\omega_z$ between atomic spin-states. To create SO-coupling, two counter-propagating $\lambda = 804.1\text{nm}$ wavelength Raman lasers, along \hat{x} -axis, are used with frequencies ω_R and $\omega_R + \delta\omega_R$, respectively and having polarization along \hat{z} -axis and \hat{y} -axis. **(b)** Energy level diagram of SO-coupled atomic spin-states $|\uparrow\rangle = |F = 1, m_F = 0\rangle$ and $|\downarrow\rangle = |F = 1, m_F = -1\rangle$, differentiated by Zeeman shift and

excited by Raman lasers having frequency difference $\delta\omega_R = \omega_z + \delta/\hbar$, where δ is the detuning between these Raman lasers. **(c)** Schematic diagram for spectral configuration of optomechanical system. When intra-cavity radiation pressure oscillates with frequency resonant to mechanical-mode frequency ($\Delta_c = \omega_c - \omega_p = \pm\omega_m$), it will generate Stokes and anti-Stokes scatterings at the edge of cavity. As we are interested in low-temperature dynamics which is only possible when system operates in resolved-sideband regime (i.e. when $\Delta_c = \omega_m$ and mechanical-mode frequency is greater than intra-cavity decay $\omega_m \gg \kappa$), therefore, only anti-Stock scatterings will survive.

Figure 2 | Eigenenergies spectrum of SO-coupled BEC. Eigenenergies spectrum E_N of spin-orbit (SO)-coupled Bose-Einstein condensate (BEC) as a function of quasi-momentum $\mathbf{k}_x/\mathbf{k}_L$ under influence of Raman coupling Ω_z/ω_m , detuning δ/ω_m and mechanical oscillator coupling with intra-cavity field g_m/ω_m . The first panel illustrates E_N , for Raman coupling Ω_z/ω_m and detuning δ/ω_m , in presence of weak mirror-field coupling $g_m = 0.1\omega_m$ and SO-coupling $\alpha = 20\pi\omega_m$. The black, blue, red, green and orange curves correspond to the Raman coupling $\Omega_z = 0\omega_m, 2\omega_m, 4\omega_m, 6\omega_m, 8\omega_m$, respectively, and indicate topological phase-transitions of atomic states from stripe, magnetized to non-magnetic phases by increasing the band-gape with increase in Raman coupling. **(a)**, **(b)** and **(c)** show the behavior of E_N for Raman detuning $\delta = -\omega_m, 0\omega_m, \omega_m$, respectively and demonstrate the asymmetric behavior of E_N for quasi-momentum with non-zero Raman detuning $\delta \neq 0\omega_m$, causing enhancement in topological-features. **(d)** and **(e)** show Eigenenergies spectrum E_N verses quasi-momentum $\mathbf{k}_x/\mathbf{k}_L$ and normalized mirror-field coupling g_m/ω_m , with SO-coupling $\alpha = 30\pi\omega_m$ and $\alpha = 50\pi\omega_m$, respectively. The Raman coupling and detuning is now fixed to $\Omega_z = \omega_m$ and $\delta = 0\omega_m$, respectively. Here, the values of eigenener-

gies are increased with increase in mirror-field coupling due to absorption of phonon emitted by mechanical oscillator of the system. In the results, normalized strength of atom-atom interactions is $U = 5.5\omega_m$, the parameter representing inter-species interactions is taken as, $\varepsilon = 0.1\omega_m$, mechanical-mode frequency is $\omega_m = 1 \times 2\pi\text{kHz}$ and intra-cavity field decay is considered as $\kappa = 0.1\omega_m$, which obeys resolved-sideband condition ($\omega_m \gg \kappa$). Further, $\gamma_a = 0.01\omega_m$ and $\gamma_m = 0.05\omega_m$ are damping rates of mechanical-mode and atomic-modes, respectively.

Figure 3 | Density-noise spectrum for atomic spin-states with spin-orbit coupling. Density-noise spectrum (DNS) $S_\uparrow(\omega, \Delta)$ and $S_\downarrow(\omega, \Delta)$ (in units of W/Hz) for atomic spin- \uparrow and spin- \downarrow states, respectively, as a function of normalized detuning Δ/ω_m and frequency ω/ω_m . **(a)** Illustrates $S_\uparrow(\omega, \Delta)$ corresponding atomic spin- \uparrow in the absence of spin-orbit (SO)-coupling $\alpha = 0\pi\omega_m$ but in the presence of Raman-coupling (or Zeeman field effects) $\Omega_z = \omega_m$ and Raman-detuning $\delta = \omega_m$. The atomic-modes coupling is taken as $G_a = 28.5\omega_m$. (Note: The color configuration corresponds to the strength of DNS.) **(b, c)** Show behavior of DNS $S_\uparrow(\omega, \Delta)$ for atomic spin- \uparrow when SO-coupling strength is increased to $\alpha = 150\pi\omega_m$ and $\alpha = 250\pi\omega_m$, respectively, while the Zeeman field effects and atom-field coupling remain the same. By increasing SO-coupling, not only the radius of semi-circular structure of $S_\uparrow(\omega, \Delta)$ decreases but also the area under $S_\uparrow(\omega, \Delta)$ is decreased which leads to the cooling of spin- \uparrow . Similarly, **(d)** demonstrates DNS $S_\downarrow(\omega, \Delta)$ for spin- \downarrow atomic-mode in the absence of SO-coupling $\alpha = 0\pi\omega_m$ and **(e)** and **(f)** show the influence of non-abelian gauge field on $S_\downarrow(\omega, \Delta)$ by increasing SO-coupling to $\alpha = 150\pi\omega_m$ and $\alpha = 250\pi\omega_m$, respectively. In the absence of SO-coupling, $S_\uparrow(\omega, \Delta)$ and $S_\downarrow(\omega, \Delta)$ behave in a similar way but the presence of SO-coupling brings difference between $S_\uparrow(\omega, \Delta)$ and $S_\downarrow(\omega, \Delta)$ due to its topological

properties. Although it decreases the radius of semi-circular structure but it enhances the area under DNS. Atom-atom interaction will influence atomic DNS in a similar. (g) Shows $S_{\uparrow}(\omega, \Delta)$ for normalized detuning Δ/ω_m and frequency ω/ω_m with atom-atom interactions strength $U = 5.5\omega_m$. The Zeeman field effects and atom-field coupling remain same while the strength of normalized SO-coupling is fixed to $\alpha = 100\pi\omega_m$. (h) and (i) illustrate $S_{\uparrow}(\omega, \Delta)$ when the strength of atom-atom interactions U/ω_m is increased to $U = 7.5\omega_m$ and $U = 9.5\omega_m$, respectively. The mirror-field coupling is $G_m = 1.5\omega_m$, the frequency of atomic-mode and mechanical-mode is $\Omega = 70.8\omega_m$ and $\omega_m = 3.8 \times 2\pi\text{kHz}$, respectively. The temperature of external heat-bath is fixed to $T = 300\text{K}$. The remaining parameters are same as in Fig. 2.

Figure 4 | Mechanical-mode density-noise spectrum. Density-noise spectrum (DNS) $S_m(\omega, \Delta)$ (in units of W/Hz) for mechanical-mode of the system verses normalized effective detuning Δ/ω_m and frequency ω/ω_m under the influence of spin-orbit (SO)-coupling α/ω_m of atomic spin-states and normalized atom-atom interactions among atomic states U/ω_m . (a) Demonstrates $S_m(\omega, \Delta)$ in the absence of SO-coupling $\alpha = 0\pi\omega_m$ and with atom-atom interactions $U = 5.5\omega_m$. The atom-field coupling is $G_a = 20\omega_m$ and mechanical-mode coupling with intra-cavity optical field is taken as $G_m = 1.5\omega_m$. The color configuration corresponds to the strength of mechanical-mode DNS ($S_m(\omega, \Delta)$). The mechanical-mode DNS show similar behavior as atomic DNS which is in the form of semi-circular structure appearing with increase in frequency and detuning. (b) Shows the behavior of mechanical-mode DNS under the influence of non-abelian gauge field \mathbf{A} inducing SO-coupling $\alpha = 60\pi\omega_m$, while, (c) illustrates the dynamics of $S_m(\omega, \Delta)$ when SO-coupling is considered as, $\alpha = 120\pi\omega_m$. The dynamics of $S_m(\omega, \Delta)$ under the effects of many-

body interactions U/ω_m are illustrated in **(d-f)**. **(d)** Shows mechanical-mode DNS as a function of normalized detuning and frequency, having atom-atom interaction $U = 5.5\omega_m$ and SO-coupling strength $\alpha = 10\pi\omega_m$. **(e)** Accommodates information about $S_m(\omega, \Delta)$ when atom-atom interactions are increased to $U = 7.5\omega_m$ and similarly, **(f)** contains behavior of mechanical DNS with interaction strength $U = 9.5\omega_m$. The SO-coupling and atom-atom interactions both influence $S_m(\omega, \Delta)$ in a similar way by decreasing the radius of semi-circular structure providing efficient control over the dynamics of mechanical-mode with SO-coupling and atom-atom interaction. The remaining parameters, used in numerical calculations, are same as in Fig. 2.

Figure 5 | Effective-temperature of mechanical-mode. The effective-temperature of mechanical-mode T_{eff} (in units of mK) verses normalized effective system detuning Δ/ω_m , under the influence spin-orbit (SO)-coupling α/ω_m , atom-field coupling G_a/ω_m and atom-atom interactions of atomic side-mode U/ω_m . The value of normalized frequency and mechanical-mode coupling with cavity field is considered as, $\omega = 0.1\omega_m$ and $G_m = 15.5\omega_m$, respectively. **(a)** Contains response of T_{eff} under different strength of SO-coupling α/ω_m when atomic side-mode coupling with intra-cavity field is kept $G_a = 20\omega_m$ and atom-atom interactions are considered $U = 5.5\omega_m$. The blue, red, green and orange curves in **(a)** correspond to the SO-coupling strength $\alpha = 0\pi\omega_m, 70\pi\omega_m, 100\pi\omega_m, 130\pi\omega_m$, respectively. **(b)** Illustrates the effective-temperature T_{eff} of mechanical-mode under the influence of atomic coupling with intra-cavity optical mode G_a/ω_m . The SO-coupling strength is now considered $\alpha = 100\pi\omega_m$ and many-body interaction are kept $U = 5.5\omega_m$. Here blue, red, green and orange curves contain the response of T_{eff} when atom-field coupling is considered as, $G_a = 23\omega_m, 26\omega_m, 29\omega_m, 40\omega_m$, respectively. Similarly, **(c)** deals with

the behavior effective-temperature T_{eff} of mechanical-mode when SO-coupling $\alpha = 100\pi\omega_m$ and atomic-modes coupling with intra-cavity field $G_a = 20\omega_m$ is kept constant and the strength of atom-atom interactions U/ω_m is changed. On the similar basis, blue, red, green and orange curves demonstrate T_{eff} of mechanical-mode when strength of atom-atom interactions are varied from $U = 5.5\omega_m, 6.5\omega_m, 7.5\omega_m, 8.5\omega_m$, respectively. The SO-coupling and atom-atom interactions both are contributing in increased effective-temperature of mechanical-mode. However, atom-field coupling is decreasing heating effects on mechanical-mode. Lower panel of figure shows the influence of Raman coupling on effective-temperature of mechanical-mode. **(d)** Shows T_{eff} as a function of SO-coupling α/ω_m and atom-atom interactions U/ω_m in the absence of Raman coupling (or topological-effects) $\Omega_z = 0\omega_m$ at detuning $\Delta = 1.5\omega_m$. While **(d)** shows mechanical-mode temperature when the Raman coupling is taken as $\Omega_z = 60\omega_m$. In absence of Raman coupling, maxima of effective-temperature appeared to be saturated with increase in SO-coupling. However, in presence of Raman coupling, it increase exponentially with increase in SO-coupling due to topological-effects. All other parameters, used in computing numerical results, are same as described in Fig. 2.

Figure 6 | Output field density-noise spectrum and dynamic structure factor. Density-noise spectrum (DNS) of optical field leaking-out of cavity $S_{out}(P, \omega)$ (in units of W/Hz) as a function of normalized input field power P/P_{cr} and normalized frequency ω/ω_m for both in presence (right) and in absence (left) of spin-orbit (SO)-coupling α/ω_m . The lower panel contains dynamic structure factor (DSF) (in units of $1/Hz$) corresponding to the out-going optical mode DNS, as a function of normalized frequency ω/ω_m , for different values of input field

power P/P_{cr} . The strength of atom-atom interaction, coupling of intra-cavity field with atomic-modes and mechanical-mode is taken as, $U = 5.5\omega_m$, $G_a = 0.9\omega_m$ and $G_m = 1.5\omega_m$, respectively. **(a)** shows $S_{out}(P, \omega)$ in the absence of SO-coupling α/ω_m while **(b)** contains the behavior of $S_{out}(P, \omega)$ under the influence of SO-coupling α/ω_m , where the strength of SO-coupling is $\alpha = 80\pi\omega_m$. The creation of two sidebands at $\omega < 0\omega_m$ and $\omega > 0\omega_m$, caused by creation and annihilation of quasi-particles, respectively, can be noted in both graphs. In both graphs, sidebands tends to approach $\omega = 0\omega_m$ with increase in power. Another feature appears in DNS, centered at $\omega = 0\omega_m$, with increase in input power corresponding to noises associated with system. The color configuration describes the strength of $S_{out}(P, \omega)$, as shown by color bar in left side of upper panel. **(c)** and **(d)** illustrate the behavior of DSF ($S_D(k, \omega)$) for normalized input field powers $P/P_{cr} = 3$ and $P/P_{cr} = 5$, respectively, in the absence of SO-coupling α/ω_m . The similar peaks can be seen associated with sidebands and central structures appearing in the out-going mode DNS. Similarly, **(e)** and **(f)** demonstrate $S_D(k, \omega)$ at input field powers $P/P_{cr} = 3$ and $P/P_{cr} = 5$, respectively, but in the presence of SO-coupling α/ω_m . The other parameter used in numerical calculations are same as used in Fig. 2.

Figure 7 | Influence of spin-orbit coupling and atom-atom interactions on dynamic structure factor. The dynamic structure factor (DSF) $S_D(k, \omega)$ (in units of $1/Hz$) as a function of normalized frequency ω/ω_m for different strengths of spin-orbit (SO)-coupling α/ω_m **(a)** and atom-atom interactions U/ω_m of atomic dressed-states **(b)**. The input field power ratio is considered as $P/P_{cr} = 4$ and the remaining coupling strengths are same as in Fig. 6. **(a)** shows $S_D(k, \omega)$ for different SO-couplings α/ω_m and the strength of atom-atom interactions are taken as

$U = 5.5\omega_m$. The blue, red, green and orange curves correspond to the SO-coupling $\alpha = 0\pi\omega_m$, $\alpha = 60\pi\omega_m$, $\alpha = 100\pi\omega_m$ and $\alpha = 140\pi\omega_m$, respectively. **(b)** illustrates $S_D(k, \omega)$ for different strength of atom-atom interactions U/ω_m of atomic dressed-states when the SO-coupling is fixed to $\alpha = 10\pi\omega_m$. Similarly, the blue, red, green and orange curves carry the influence of atom-atom interaction, with strengths $U = 5.5\omega_m$, $U = 7.5\omega_m$, $U = 9.5\omega_m$ and $U = 11.5\omega_m$, respectively, on $S_D(k, \omega)$ of the system. By analyzing the results, one can observe that the SO-coupling α/ω_m and atom-atom interactions U/ω_m both are suppressing the noise effects and enhancing the secondary peak in $S_D(k, \omega)$ corresponding to the sidebands. The other parameters used in calculations of these results are same as discussed in Fig. 2.













